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A DISCUSSION OF STRESS RATES IN FINITE DEFORMATION PROBLEMS†

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Abstract—It has recently been shown that the finite elastic–plastic solution of the simple shear problem exhibits an oscillatory stress response when kinematic hardening is employed, while the solution for isotropic hardening gives a monotonically increasing stress. This paper analyzes this response on the basis of continuum mechanical descriptions of the problem. Three objective stress rates are recalled and spatial descriptions of plasticity at finite deformation are reviewed for the usual generalization of the infinitesimal theory as well as a theory based on an invariant measure of true stress. In light of the equations for the evolution of the yield surface, the hypoelastic solution to the simple shear problem for each of the three stress rates is presented. It is shown that the use of the Jaumann rate in the generalization of the infinitesimal theory leads to an oscillation in the evolution of the yield surface in simple shear which is explained on the basis of the hypoelastic solution. An alternative theory which makes use of the polar decomposition predicts a monotonically increasing shear stress.

INTRODUCTION

An interesting difficulty in the numerical evaluation of the large deformation simple-shear problem has recently been presented by Nagtegaal and De Jong[1]. The trouble encountered in [1] is that when kinematic hardening of the plastic deformation is implemented for a material which strain hardens monotonically in tension, the resulting shear stress exhibits oscillatory response with increasing shear strain. It is also reported in [1] that this difficulty does not arise if isotropic hardening is employed.

This paper summarizes the results of a study whose purpose is to explain the phenomena observed in [1] and to put them into proper perspective regarding established concepts in continuum mechanics. It follows the work of Lee *et al.*[2] who discuss the effect on the basis of finite deformation plasticity and propose a new objective stress rate to avoid the oscillatory response found in [1]. Lee's solution[2] depends on the introduction of a particular hardening mechanism (kinematic hardening), whereas we show that the response is actually more fundamentally associated with the rate type constitutive model and that it can occur without consideration of plasticity at all.

We begin by reviewing the basic relations between the kinematic and stress variables at finite deformation, including a discussion of their rates and restrictions imposed by the requirements of invariance under superposed rigid body motion. Use is made of the polar decomposition of the deformation gradient to introduce a set of kinematic and stress quantities which have the convenient property of being unchanged under superposed rigid body motions. Next we recall three measures of the rate of Cauchy (true) stress and introduce the hypoelastic constitutive equation in general form. Following this, we discuss two theories of plasticity at finite deformation—one which employs the polar decomposition and the other which is the usual extension of infinitesimal plasticity which led to

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the oscillatory response noted in [1]. We pay particular attention to the constitutive equations for the hardening variables in the cases of isotropic and kinematic hardening. In this we show that there is a fundamental difference in the form of these equations between the two cases. Specificially we show that for isotropic hardening an invariant scalar variable controls the hardening, while for kinematic hardening the constitutive equation for the tensor hardening variable has the form of the hypoelastic constitutive relation. With this in mind we consider analytical solutions to the simple shear problem using a hypoelastic constitutive equation with the three stress rates introduced previously. The hypoelastic solution using the rate employed in the usual extension of infinitesimal plasticity gives a shear stress which is a sinusoidal function of the angle of shear. The other stress rates lead to monotonically increasing shear stresses as the deformation proceeds. We conclude that the usual extension of plasticity predicts the oscillatory response of the simple shear problem with kinematic hardening because of the particular stress rate that is employed and that any theory (not only plasticity) which uses that stress rate is similarly subject to such response.†

STRESS AND STRAIN AT FINITE DEFORMATION

Let us identify a material point X in a reference configuration B_0 by the position vector $\mathbf{X}$ and in the current state of deformation B by the vector $\mathbf{x}$, where $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. The deformation gradient $\mathbf{F}$ associated with the motion between B_0 and B is defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \qquad J = \det \mathbf{F} > 0. \tag{1}$$

Recalling the polar decomposition theorem, we can write

$$\mathbf{F} = \mathbf{R}\mathbf{U} \tag{2}$$

where $\mathbf{U}$ is a symmetric, positive-definite tensor whose principal values are the stretch ratios of the deformation, and $\mathbf{R}$ is a proper orthogonal tensor called the local rotation tensor.

Associated with the polar decomposition (2) is an intermediate configuration $\bar{B}$ which is attained after the stretch $\mathbf{U}$, but before the rotation $\mathbf{R}$. In this form of the polar decomposition, an arbitrary line element $d\mathbf{X}$ in B_0 is stretched by $\mathbf{U}$ into $d\boldsymbol{\xi}$ in $\bar{B}$, and is subsequently rotated by $\mathbf{R}$ into $d\mathbf{x}$ in B. We note here two important features of the decomposition: First, the local rotation between $\bar{B}$ and B involves no stretching, and second, in addition to the rotation $\mathbf{R}$, the stretch tensor $\mathbf{U}$ causes a rotation of any line element which is not aligned with the principal directions of $\mathbf{U}$.

The Lagrange strain $\mathbf{E}$ is

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}), \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \tag{3}$$

where $\mathbf{I}$ is the identity tensor and $\mathbf{C}$ is the right Cauchy–Green tensor.

The velocity of point X is $\mathbf{v} = \dot{\mathbf{x}}$ where superposed dot indicates material time derivative holding $\mathbf{X}$ fixed. The associated velocity gradient is

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{4}$$

The symmetric part $\mathbf{D}$ of the velocity gradient is called the rate of deformation tensor while the skew-symmetric part $\mathbf{W}$ is called the spin or vorticity tensor. We also define for later use

†The association of the problem with stress rates was first suggested to the authors by R. D. Krieg and S. W. Key.

a rate of rotation tensor $\boldsymbol{\Omega}$ and an unrotated rate of deformation tensor $\mathbf{d}$ as

$$\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T, \qquad \mathbf{d} = \mathbf{R}^T\mathbf{D}\mathbf{R} = \frac{1}{2}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}) \tag{5}$$

where $\boldsymbol{\Omega}$ is skew symmetric since $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ and represents the angular velocity associated with the local rotation $\mathbf{R}$. The rate of strain is related to the rates of deformation as

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F} = \mathbf{U}\mathbf{d}\mathbf{U}. \tag{6}$$

In discussing the stress and stress rate at large deformation, we will examine three different measures—the symmetric Piola–Kirchhoff stress $\mathbf{S}$, the Cauchy (or true) stress $\mathbf{T}$ and an unrotated Cauchy stress $\boldsymbol{\sigma}$. These stress measures are related as

$$\mathbf{T} = \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T, \qquad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{U}\mathbf{S}\mathbf{U} = \mathbf{R}^T\mathbf{T}\mathbf{R}. \tag{7}$$

In light of our discussion of the polar decomposition, we note that the unrotated Cauchy stress is the "true" stress associated with the stretch $\mathbf{U}$ alone. That is, it is the true stress referred to the intermediate configuration $\bar{\mathscr{B}}$. Further, since $\mathbf{R}$ is a proper orthogonal tensor the principal invariants of $\mathbf{T}$ and $\boldsymbol{\sigma}$ are identical.

We can now recall some basic results concerning invariance under superposed rigid-body motions. Thus, consider a motion which differs from the given motion only by a superposed rigid body motion, i.e.

$$\mathbf{x}^+ = \mathbf{a}(t) + \mathbf{Q}(t)\mathbf{x} \tag{8}$$

where $\mathbf{a}(t)$ represents a rigid-body translation, and $\mathbf{Q}(t)$ is a proper orthogonal tensor representing a rigid-body rotation of the deformed state.

For quantities associated with the motion (8) we use the same symbol, but add a superscript plus (+). The basic kinematic variables transform as

$$\mathbf{F}^+ = \mathbf{Q}\mathbf{F}, \qquad \mathbf{R}^+ = \mathbf{Q}\mathbf{R}, \qquad \mathbf{U}^+ = \mathbf{U}, \qquad \mathbf{C}^+ = \mathbf{C}, \qquad \mathbf{E}^+ = \mathbf{E} \tag{9}$$

while the rate relations are

$$\begin{gathered} \mathbf{L}^+ = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\mathbf{L}\mathbf{Q}^T, \qquad \mathbf{D}^+ = \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \qquad \mathbf{W}^+ = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\mathbf{W}\mathbf{Q}^T \\ \boldsymbol{\Omega}^+ = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\boldsymbol{\Omega}\mathbf{Q}^T, \qquad \mathbf{d}^+ = \mathbf{d}, \qquad \dot{\mathbf{E}}^+ = \dot{\mathbf{E}}. \end{gathered} \tag{10}$$

Also, the various stresses and their rates transform as

$$\mathbf{S}^+ = \mathbf{S}, \qquad \mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T, \qquad \boldsymbol{\sigma}^+ = \boldsymbol{\sigma} \tag{11}$$

and

$$\dot{\mathbf{S}}^+ = \dot{\mathbf{S}}, \qquad \dot{\boldsymbol{\sigma}}^+ = \dot{\boldsymbol{\sigma}}, \qquad \dot{\mathbf{T}}^+ = \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T. \tag{12}$$

On examination of these transformations, we observe that certain of the rate variables, in particular $\mathbf{L}$, $\mathbf{W}$, $\boldsymbol{\Omega}$ and $\dot{\mathbf{T}}$, are not objective in that they are neither unchanged nor unchanged apart from orientation under superposed rigid body motion.† The implications of this fact are discussed below in relation to the derivation of an appropriate stress rate.

†The terminology used here is that of [3] in which a more complete discussion of invariance under superposed rigid body motions is to be found.

STRESS RATE

In many circumstances, the description of elastic–plastic response, for example, it is desirable to write the constitutive relation for a material in terms of rate of stress and rate of strain or deformation. Often we want to relate the rate of change of deformation as measured in configuration B to the rate of change of stress measures in that configuration (the rate of Cauchy stress). Unfortunately, we see from eqn (12) that $\dot{\mathbf{T}}$ is not an objective quantity and hence cannot be used alone in a constitutive equation. It is this lack of objectivity of $\dot{\mathbf{T}}$ which has prompted the introduction of the various stress rates which have been discussed in the literature and have been implemented in the computer codes for large deformation problems.

The problem addressed here has been considered by numerous authors[4–8] and has been approached from a variety of directions. It should be noted at the outset that a sizable body of the mechanics community has regarded the discussion of stress rates as a dead issue[9–11]. The justification for this feeling lies in the theory of hypoelasticity in which a rate type constitutive equation is defined to be of the form

$$\mathring{\mathbf{T}} = f(\mathbf{T}, \mathbf{D}) \tag{13}$$

where $\mathring{\mathbf{T}}$ represents any measure of the rate of change of Cauchy stress which is unchanged apart from orientation under superposed rigid body motions. All of the stress rates considered in hypoelasticity may be thought of as being equivalent to one another provided the r.h.s. of (13) is appropriately chosen for each measure. In the following paragraphs we discuss the three objective stress rates introduced by Jaumann[12], Truesdell[13], and Green and Naghdi[14, 15]. While there have been many more stress rates proposed these are chosen because of the wide use of the Jaumann rate, the natural connection between the Truesdell rate and nonlinear elasticity, and the association of the Green–Naghdi rate with the definition of a simple material (in the sense of Noll[16]).

Jaumann's definition, denoted $\mathring{\mathbf{T}}^a$, creates an objective rate quantity from $\dot{\mathbf{T}}$ by introducing the form

$$\mathring{\mathbf{T}}^a = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}. \tag{14}$$

This particular stress rate (which is also called the co-rotational stress rate), is currently the most popular of the many available. The preference of eqn (14) over the others seems to stem from a paper by Prager[5] in which it is shown that if $\mathring{\mathbf{T}}^a$ is zero for some motion, then the invariants of the Cauchy stress are stationary during the motion. Unfortunately, this measure has been found to lead to a nonsymmetric stiffness tensor between rates of stress and deformation unless the material is incompressible[17].† In addition, Bazant[8] indicates that while the Jaumann rate is objective, it has no conjugate measure of finite strain associated with it.

Truesdell's stress rate, $\mathring{\mathbf{T}}^b$, may be obtained by differentiating the expression for $\mathbf{T}$ in terms of $\mathbf{S}$, eqn (7), giving

$$\mathring{\mathbf{T}}^b = \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T + \mathbf{T}\,(\mathrm{tr}\,\mathbf{D}) = \frac{1}{J}\,\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^T. \tag{15}$$

From this we see that the Truesdell stress rate is just the expression for the rate of change of the symmetric Piola–Kirchhoff stress, expressed in the deformed configuration B. Thus, vanishing of $\mathring{\mathbf{T}}^b$ implies that $\mathbf{S}$ is constant during the motion. This stress rate would naturally appear in problems in which a strain energy function gives the constitutive equation between quantities in the reference configuration B_0, the symmetric Piola–Kirchhoff stress $\mathbf{S}$ and the Lagrange strain $\mathbf{E}$. In cases when one is trying to obtain

† This lack of symmetry can be avoided by introducing the Kirchhoff stress, $\tau = J\mathbf{T}$, and defining the constitutive equation in terms of the Jaumann rate of τ, $\mathring{\tau}^a = \dot{\tau} - \mathbf{W}\tau + \tau\mathbf{W}$.

a constitutive relation between stress rate in the deformed configuration B and the rate of deformation, $\hat{\mathbf{T}}_b$ may not be as useful since neither $\mathbf{T}$ nor its invariants are stationary for vanishing $\hat{\mathbf{T}}_b$.

The final stress rate, denoted $\hat{\mathbf{T}}^c$, is obtained by differentiating the expression for $\mathbf{T}$ in terms of $\boldsymbol{\sigma}$ giving

$$\hat{\mathbf{T}}^c = \dot{\mathbf{T}} - \boldsymbol{\Omega}\mathbf{T} + \mathbf{T}\boldsymbol{\Omega} = \mathbf{R}\dot{\boldsymbol{\sigma}}\mathbf{R}^T. \tag{16}$$

We see that this stress rate represents the derivative of an invariant measure of stress, $\boldsymbol{\sigma}$, whose principal invariants have the same values as those for the Cauchy stress. Thus, if $\hat{\mathbf{T}}^c$ is zero, Prager's condition that the invariants of the Cauchy stress be stationary [5] is satisfied. A possible advantage of this rate measure over Jaumann's is that it is derivable from a physically meaningful stress, $\boldsymbol{\sigma}$, and is truly a measure of the rate of change of stress in the deformed configuration.

The use of $\boldsymbol{\sigma}$ is an appropriate measure for stress in a spatial description of motion is certainly not new. In fact, it is this stress measure which is used in the definition of a simple material [10, 16]. The definition of a simple material is one in which the stress $\boldsymbol{\sigma}$ may be written as a functional of the stretch tensor $\mathbf{U}$ alone. Noll [16], in his derivation of the simple material, defines a rate of stress by taking a derivative of $\boldsymbol{\sigma}$, but his use of the relative deformation gradient leads him to a stress rate which is equivalent to the Jaumann rate.

ELASTIC–PLASTIC RESPONSE AT FINITE DEFORMATION

With the basic concepts of the previous two sections in mind, we begin our discussion of the theory of plasticity at finite deformation. In this section we examine the spatial formulation of two finite deformation plasticity theories: that of Green and Naghdi [14] and the usual generalization of the infinitesimal plasticity [1, 18–20]. In the former derivation, all kinematic and stress quantities are defined with respect to the intermediate configuration $\bar{B}(\mathbf{d}, \boldsymbol{\sigma}, \text{etc})$ while in the latter, all such quantities are referred to the current configuration $B(\mathbf{D}, \mathbf{T}, \text{etc.})$.

Motivated by the simple shear response observed in [1], we consider in detail the forms of the equations for the two theories for the cases of ideal isotropic and kinematic hardening with a Mises yield criterion. It is shown that a fundamental difference in the structure of the equation for the two hardening mechanisms accounts for the difference in the solutions noted in [1].

We will discuss the theory of plasticity in terms of a yield function or yield surface in stress space. We consider all motions which are "inside" the yield surface to be elastic motions and those which are on the surface itself to be elastic–plastic motions. From a point on the yield surface there are three possibilities: loading if the increment of additional stress is directed outward from the surface, unloading if the increment of additional stress is directed inward, and neutral loading if the stress increment is tangent to the surface. The question of hardening arises in relation to the evolution of the yield surface as loading proceeds.

Following the derivation in [14], the rate of deformation is decomposed into elastic and plastic parts as

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \tag{17}$$

and the unrotated rate of deformation $\mathbf{d}$ is decomposed as

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p \tag{18}$$

where

$$\mathbf{d}^e = \mathbf{R}^T\mathbf{D}^e\mathbf{R}, \qquad \mathbf{d}^p = \mathbf{R}^T\mathbf{D}^p\mathbf{R}. \tag{19}$$

For our discussion of isotropic and kinematic hardening the yield surface in the spatial description will be written in terms of quantities referred to the intermediate configuration $\bar{B}$ as

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \kappa \tag{20}$$

where $\boldsymbol{\alpha}$ is the kinematic hardening variable and κ is the scalar work-hardening parameter associated with isotropic hardening. The second-order tensor $\boldsymbol{\alpha}$ is taken to have units of stress and to have the same invariance properties as $\boldsymbol{\sigma}$. We refer to $\boldsymbol{\alpha}$ as the back-stress. For fixed values of $\boldsymbol{\alpha}$ and κ, eqn (20) represents a surface in six-dimensional stress space.

The constitutive equations for the rates of change of the plastic strain, back-stress and work-hardening may be expressed as

$$d^p = \begin{cases} 0 & f < \kappa, \\ 0 & f = \kappa, \quad \hat{f} < 0 \\ 0 & f = \kappa, \quad \hat{f} = 0 \\ \boldsymbol{\beta}\hat{f} & f = \kappa, \quad \hat{f} > 0 \end{cases}$$

$$\dot{\boldsymbol{\alpha}} = \mathbf{g}\mathbf{d}^p, \qquad \dot{\kappa} = \text{tr}\,\{\mathbf{h}\mathbf{d}^p\} \tag{21}$$

where $\hat{f}$ is the loading index,

$$\hat{f} = \text{tr}\left\{\frac{\partial f}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}\right\}, \tag{22}$$

$\boldsymbol{\beta}$ and $\mathbf{h}$ are second-order tensor functions of $\boldsymbol{\sigma}$, α and κ, and $\mathbf{g}$ is a fourth-order tensor function of the same arguments. The four conditions involving f and $\hat{f}$ in the first of eqn (21) represent elastic deformation, unloading, neutral loading and loading from an elastic–plastic state, respectively.

The difficulty associated with modelling an actual material lies in the specification of the yield function f and the constitutive tensor functions $\boldsymbol{\beta}$, $\mathbf{h}$ and $\mathbf{g}$. The usual approach for establishing the form of $\boldsymbol{\beta}$ is to apply the condition that the plastic increment of the deformation is in a direction perpendicular to the yield surface. For the infinitesimal theory this normality rule can be derived from Drucker's postulate that the work done during any cyclic motion be non-negative. For a finitely deformed material, however, Naghdi and Trapp[21] have shown that an assumption that the work done in a cyclic motion is non-negative does not necessarily lead to a normality condition. Recognizing this, it is sufficient for our purposes to assume that the material is such that normality holds. We now proceed to discuss hardening in the extremely idealized cases of isotropic and kinematic hardening.

In discussing isotropic hardening, we use the Mises yield criterion

$$f = \frac{1}{2}\,\text{tr}\,\{\boldsymbol{\sigma}'^2\}, \qquad \kappa = \frac{Y^2}{3}, \qquad \boldsymbol{\alpha} = \dot{\boldsymbol{\alpha}} = 0 \tag{23}$$

where $\boldsymbol{\sigma}'$ is the deviatoric part of the stress,

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}\,\text{tr}\,\{\boldsymbol{\sigma}\}\,\mathbf{I} \tag{24}$$

and Y is the yield stress in tension. The assumed normality condition implies that $\boldsymbol{\beta}$ in eqn (21) be a scalar multiple of $\boldsymbol{\sigma}'$ so that

$$\mathbf{d}^p = \gamma_I \hat{f}\boldsymbol{\sigma}', \qquad \hat{f} = \text{tr}\,\{\sigma'\dot{\sigma}\} = \dot{\kappa}. \tag{25}$$

If we introduce an effective plastic strain rate

$$\dot{\bar{\epsilon}}^p = \left[\frac{2}{3}\operatorname{tr}\{\mathbf{d}^{p^2}\}\right]^{1/2} \tag{26}$$

and a corresponding effective plastic strain as

$$\bar{\epsilon}^p = \int_{t_0}^{t} \dot{\bar{\epsilon}}^p \, \mathrm{d}t \tag{27}$$

then we can take the yield stress to be written as

$$Y = Y(\bar{\epsilon}^p). \tag{28}$$

Then, from eqn (23) we have that

$$\dot{\kappa} = \frac{2}{3} Y\dot{Y} = \frac{2}{3} Yh\dot{\bar{\epsilon}}^p \tag{29}$$

where the hardening modulus h is obtained from the equation

$$h = \frac{\mathrm{d}Y}{\mathrm{d}\bar{\epsilon}^p}. \tag{30}$$

The plastic strain rate from eqn (25) can be used in eqn (26) to write $\dot{\bar{\epsilon}}^p$ as

$$\dot{\bar{\epsilon}}^p = \frac{2}{3}\gamma_I \hat{f} Y. \tag{31}$$

However, since $\hat{f} = \dot{\kappa}$ and $\dot{\kappa}$ is given in eqn (29) as a function of $\dot{\bar{\epsilon}}^p$, we can solve for γ_I in terms of h as

$$\gamma_I = \frac{9}{4hY^2} \tag{32}$$

so that

$$\mathbf{d}^p = \frac{9}{4hY^2}\operatorname{tr}\{\boldsymbol{\sigma}'\dot{\boldsymbol{\sigma}}\}\boldsymbol{\sigma}'. \tag{33}$$

Thus we see that the isotropic hardening evolves through the scalar equation for $\dot{\kappa}$ which is proportional to the hardening modulus h.

We turn now to kinematic hardening, in which the evolution in the yield surface occurs not by its changing size, but by translation of its center with respect to the unstressed state. To effect this change in position, the back stress $\boldsymbol{\alpha}$ is employed and the yield function is written

$$f = \frac{1}{2}\operatorname{tr}\{(\boldsymbol{\sigma}' - \boldsymbol{\alpha})^2\}, \qquad \kappa = \frac{Y_0^2}{3}, \qquad \dot{\kappa} = 0 \tag{34}$$

where Y_0 is the initial yield stress in uniaxial tension. The normality condition now reqiures that $\boldsymbol{\beta}$ be proportional to $\boldsymbol{\sigma}' - \boldsymbol{\alpha}$ so

$$\mathbf{d}^p = \gamma_k \hat{f}(\boldsymbol{\sigma}' - \boldsymbol{\alpha}), \qquad \hat{f} = \operatorname{tr}\{(\boldsymbol{\sigma}' - \boldsymbol{\alpha})\dot{\boldsymbol{\sigma}}\}. \tag{35}$$

Since $\dot{\kappa} = 0$, evolution of the yield surface is given by the evolution of $\boldsymbol{\alpha}$, eqn (21). The classical assumption associated with kinematic hardening is that the hardening variable $\boldsymbol{\alpha}$ moves in the direction of plastic flow, so that

$$\dot{\boldsymbol{\alpha}} = \delta_k \gamma_k \operatorname{tr}\{(\boldsymbol{\sigma}' - \boldsymbol{\alpha})\dot{\boldsymbol{\sigma}}\}(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \tag{36}$$

where δ_k is a scalar. If we multiply both sides of eqn (36) by $(\boldsymbol{\sigma}' - \boldsymbol{\alpha})$ and take the trace we find that

$$\gamma_k = \frac{3}{2\delta_k Y_0^2} \tag{37}$$

where we have used the fact that $\dot{f} = \dot{\kappa} = 0$. Again introducing the hardening modulus h, we find from the consideration of a uniaxial tension test that

$$\gamma_k = \frac{2}{3} h \tag{38}$$

so that

$$\begin{aligned} \mathbf{d}^p &= \frac{9}{4hY_0^2} \operatorname{tr}\{(\boldsymbol{\sigma}' - \boldsymbol{\alpha})\dot{\boldsymbol{\sigma}}\}(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \\ \dot{\boldsymbol{\alpha}} &= \frac{2}{3} h \mathbf{d}^p. \end{aligned} \tag{39}$$

If eqn $(39)_2$ is premultiplied by $\mathbf{R}$ and postmultiplied by $\mathbf{R}^T$, it is clear that the constitutive equation for $\boldsymbol{\alpha}$ has the same form as the fundamental constitutive equation for hypoelasticity, eqn (13), using $\hat{\mathbf{T}}^c$. Thus the evolution of kinematic hardening is through a hypoelastic type of equation while the evolution of isotropic hardening is through the growth of the scalar parameter κ.

Let us now consider the parallel development of a plasticity theory involving kinematic and stress measures in the current configuration B. It is such a theory that was used in [1] and that led to such different responses in the two cases of hardening. We again assume that the additive decomposition of the velocity gradient is valid, eqn (17), and let the yield surface be written as

$$f^*(\mathbf{T}, \boldsymbol{\alpha}^*) = \kappa^* \tag{40}$$

where $\boldsymbol{\alpha}^*$ and κ^* are again the kinematic and isotropic hardening variables, with $\boldsymbol{\alpha}^*$ having units of stress and exhibiting the same response under superposed rigid body motion as $\mathbf{T}$. As noted in [14], the requirement that the value of κ^* be unchanged under superposed rigid body motions restricts f^* to be an isotropic function in both $\mathbf{T}$ and $\boldsymbol{\alpha}^*$. In addition, we recall that the material time derivative of $\mathbf{T}$ (and hence also of $\boldsymbol{\alpha}^*$) is not objective and so we must introduce one of the various stress rates discussed earlier. The rate usually chosen for this is the Jaumann rate[1, 18–20].

The constitutive relations for the plastic deformation and hardening variables are then

$$\mathbf{D}^p = \begin{cases} 0, & f^* < \kappa^*, \\ 0, & f^* = \kappa^*, \quad \hat{f}^* < 0 \\ 0, & f^* = \kappa^*, \quad \hat{f}^* = 0 \\ \boldsymbol{\beta}^* \hat{f}^*, & f^* = \kappa^*, \quad \hat{f}^* > 0 \end{cases}$$

$$\hat{\boldsymbol{\alpha}}^* = g^* \mathbf{D}^p, \qquad \kappa^* = \operatorname{tr}\{\mathbf{h}^* \mathbf{D}^p\} \tag{41}$$

where the loading index is

$$\hat{f}^* = \mathrm{tr}\left\{\frac{\partial f^*}{\partial \mathbf{T}}\hat{T}^a\right\} \tag{42}$$

and $\hat{\boldsymbol{\alpha}}^*$ is the Jaumann rate of the back-stress $\boldsymbol{\alpha}^*$

$$\hat{\boldsymbol{\alpha}}^* = \dot{\boldsymbol{\alpha}} - \mathbf{W}\boldsymbol{\alpha} + \boldsymbol{\alpha}\mathbf{W}. \tag{43}$$

The case of isotropic hardening, using the Mises yield criterion, has the yield function written as

$$f^* = \frac{1}{2}\mathrm{tr}\,\{\mathbf{T}'^2\}, \qquad \kappa^* = \frac{Y^2}{3}, \qquad \boldsymbol{\alpha}^* = \hat{\boldsymbol{\alpha}}^* = 0 \tag{44}$$

where Y has the same meaning as before and $\mathbf{T}'$ is the deviatoric part of the Cauchy stress

$$\mathbf{T}' = \mathbf{T} - \frac{1}{3}\mathrm{tr}\,\{\mathbf{T}\}\mathbf{I}. \tag{45}$$

Normality implies that the plastic strain rate is

$$\mathbf{D}^p = \gamma_I^* \hat{f}^* \mathbf{T}', \qquad \hat{f}^* = \mathrm{tr}\,\{\mathbf{T}'\mathbf{T}^a\} = \dot{\kappa}^* \tag{46}$$

where we have used the fact that

$$\mathrm{tr}\,\{\mathbf{T}'\dot{\mathbf{T}}\} = \mathrm{tr}\,\{\mathbf{T}'\hat{\mathbf{T}}^a\}. \tag{47}$$

Since $\mathbf{d}^p = \mathbf{R}^T\mathbf{D}^p R$, the expression for the effective plastic strain rate, eqn (26), can be rewritten

$$\dot{\bar{\epsilon}}^p = \left[\frac{2}{3}\mathrm{tr}\,\{\mathbf{D}^{p2}\}\right]^{1/2}. \tag{48}$$

Using the same procedure as used with the theory of Green and Naghdi [14], we can show that

$$\dot{\kappa}^* = \frac{2}{3}Yh\dot{\bar{\epsilon}}^p, \qquad \mathbf{D}^p = \frac{9}{4hY^2}\mathrm{tr}\,\{\mathbf{T}'\hat{\mathbf{T}}^a\}\mathbf{T}'. \tag{49}$$

Recalling eqns (29) and (33), any noting that $\boldsymbol{\sigma}' = \mathbf{R}^T\mathbf{T}'\mathbf{R}$, we find that the constitutive equations for isotropic hardening obtained from the two theories are identical.

For the case of kinematic hardening we let

$$f^* = \frac{1}{2}\mathrm{tr}\,\{(\mathbf{T}' - \boldsymbol{\alpha}^*)^2\}, \qquad \kappa^* = \frac{Y_0^2}{3}, \qquad \dot{\kappa}^* = 0. \tag{50}$$

Normality implies that

$$\mathbf{D}^p = \gamma_k^* \hat{f}^*(\mathbf{T}' - \boldsymbol{\alpha}^*), \qquad \hat{f}^* = \mathrm{tr}\,\{(\mathbf{T}' - \boldsymbol{\alpha}^*)\hat{\mathbf{T}}^a\} \tag{51}$$

and the classical hardening assumption for $\boldsymbol{\alpha}^*$ implies that

$$\hat{\boldsymbol{\alpha}}^* = \delta_k^*\mathbf{D}^p. \tag{52}$$

Again considering the uniaxial test we find that

$$\mathbf{D}^{*p} = \frac{9}{4hY_0^2} \operatorname{tr}\{(\mathbf{T}' - \boldsymbol{\alpha}^*)\hat{\mathbf{T}}^a\}(\mathbf{T}' - \boldsymbol{\alpha}^*), \qquad \hat{\boldsymbol{\alpha}}^* = \frac{2}{3} h \mathbf{D}^p. \tag{53}$$

On comparing eqns (39) and (53), we find that the expressions for the rate of plastic deformation are the same, but since $\hat{\boldsymbol{\alpha}}^* \neq \mathbf{R}\dot{\boldsymbol{\alpha}}\mathbf{R}^T$, the expressions for the rate of change of the back-stress are different. Thus, while the two approaches give the same result for the case of isotropic hardening, they do not for the case of kinematic hardening. As we show in the next section, the oscillatory response of the simple-shear problem with kinematic hardening is due to the evolution of $\boldsymbol{\alpha}^*$ as governed by the rate equation (53).

SIMPLE SHEAR-HYPOELASTIC SOLUTIONS

Simple shear is a deformation controlled process in which the stress is evaluated on the basis of suitable constitutive equations. In solving this problem, our choice for the constitutive equations is guided by the hypoelastic equations governing the evolution of the back-stress, eqns (39) and (53). The general form of this simplest of hypoelastic relations is

$$\hat{\mathbf{T}} = \mathbf{q}(\mathbf{D}) \tag{54}$$

where $\mathbf{q}$ is a tensor function which is linear of degree one in $\mathbf{D}$, and $\hat{\mathbf{T}}$ represents either $\hat{\mathbf{T}}^a$, $\hat{\mathbf{T}}^b$ or $\hat{\mathbf{T}}^c$. While eqn (54) would at first appear to be rather general in form, we note that the behavior of $\hat{\mathbf{T}}$ and $\mathbf{D}$ under superposed rigid body motions restricts $\mathbf{q}$ to be an isotropic tensor function in $\mathbf{D}$. Thus we may write

$$\hat{\mathbf{T}} = \lambda \operatorname{tr} \mathbf{D}\, \mathbf{I} + 2\mu\mathbf{D}. \tag{55}$$

Solutions of the simple shear problem at finite deformation using eqn (55) with $\hat{\mathbf{T}}^a$ and $\hat{\mathbf{T}}^b$ may be found in [9], while the use of $\hat{\mathbf{T}}^c$ in eqn (55) was considered by Dienes[22]. We recall these solutions here and show how they relate to the hardening equations of the previous section.

At any time, t, the deformation associated with simple shear is given in terms of the current position as

$$x_1(t) = X_1 + k(t)X_2, \qquad x_2(t) = X_2 \qquad x_3(t) = X_3. \tag{56}$$

The associated deformation and velocity gradients are then

$$\mathbf{F} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} 0 & \dot{k} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{57}$$

Evaluation of the stress response for the three rates consists of using eqn (57) in the appropriate stress rate equations and then solving the resulting differential equations. For the Jaumann rate it is a simple matter to show that

$$\dot{T}^a_{11} - \dot{k}T^a_{12} = 0, \qquad \dot{T}^a_{12} + \frac{1}{2}\dot{k}\,(T^a_{11} - T^a_{22}) = \dot{k}\mu, \qquad \dot{T}^a_{22} + \dot{k}T^a_{12} = 0 \tag{58}$$

whose solution is

$$T^a_{11} = -T^a_{22} = \mu(1 - \cos k), \qquad T^a_{12} = \mu \sin k. \tag{59}$$

On examination, we see that the hypoelastic solution employing the Jaumann rate exhibits a sinusoidal response for the applied shear stress. Recall that the evolution equation for the kinematic hardening variable α^*, eqn (53), is a hypoelastic type equation using the Jaumann rate. Thus, we find that for constant hardening modulus h the shear stress component of the back stress should vary as a sinusoid and the oscillatory response observed in [1] is related directly to the hypoelastic solution, eqn (59).

Having shown the cause of the oscillatory response of the simple shear solution when using the usual plasticity theory which employs the Jaumann rate, we turn our attention to the response of the same problem when the plasticity theory of Green and Naghdi [14] is used. In this case, the response of the back-stress α is governed by a hypoelastic equation involving the stress rate $\hat{\mathbf{T}}^c$.

In examining the simple shear response using $\hat{\mathbf{T}}^c$, we refer to the analysis of Dienes [22] in which he shows that†

$$\begin{aligned} T^c_{11} &= -T^c_{22} = 4\mu\,(\cos 2\beta \ln(\cos\beta) + \beta \sin 2\beta - \sin^2\beta) \\ T^c_{12} &= 2\mu \cos 2\beta(2\beta - 2\tan 2\beta \ln(\cos\beta) - \tan\beta) \end{aligned} \tag{60}$$

where

$$\beta = \tan^{-1}(k/2). \tag{61}$$

In this case, the shear stress in a monotonically increasing function of k. Thus, we would not expect to find an oscillatory response in the back-stress when using the plasticity theory of Ref. [14].

A final stress rate, the Truesdell rate, is also considered. For this case we find the equations

$$\dot{T}^b_{11} - 2\dot{k}T^b_{12} = 0, \qquad \dot{T}^b_{12} - \dot{k}T^b_{22} = \dot{k}\mu, \qquad \dot{T}^b_{22} = 0 \tag{62}$$

which may be integrated to give

$$T^b_{12} = \mu k, \qquad T^b_{11} = \mu k^2, \qquad T^b_{22} = 0. \tag{63}$$

This response does not exhibit sinusoidal behavior, but as noted before is the result of an expression for the rate of change of the symmetric Piola–Kirchhoff stress. We again note that the vanishing of $\hat{\mathbf{T}}^b$ does not ensure that the invariants of the Cauchy stress are constants. As such, it is difficult to establish a meaningful loading index in terms of $\hat{\mathbf{T}}^b$ and its use in an elastic–plastic theory has been limited.

DISCUSSION

We consider now a numerical example in which the elastic–plastic simple shear problem is solved for three cases of hardening: isotropic hardening, eqn (33) or (49); kinematic hardening using the theory of Green and Naghdi [14], eqn (39); and kinematic hardening using the usual plasticity theory of [1, 18–20], eqn (53). In all cases, the hardening response is bi-linear in uniaxial tension, with initial yield at 345 MPa and a hardening modulus of 138 MPa. The elastic Youngs modulus is 206 GPa with Poisson's ratio of 0.33. The plots of shear stress T_{12} versus shear strain k as determined by the GNATS2 computer program [23] are shown in Fig. 1.

The isotropic hardening exhibits bilinear response at a level below the uniaxial data by a factor of $\sqrt{3}$ in accordance with the Mises yield criterion used. The kinematic hardening using the Jaumann rate exhibits a sinusoidal response beyond the yield point as we had anticipated following eqns (53) and (59). The response for the kinematic

† Note that in the expression for the shear stress, the term $\tan 2\beta$ was incorrectly printed $\tan^2\beta$ in [22] (see eqn (5.25)).

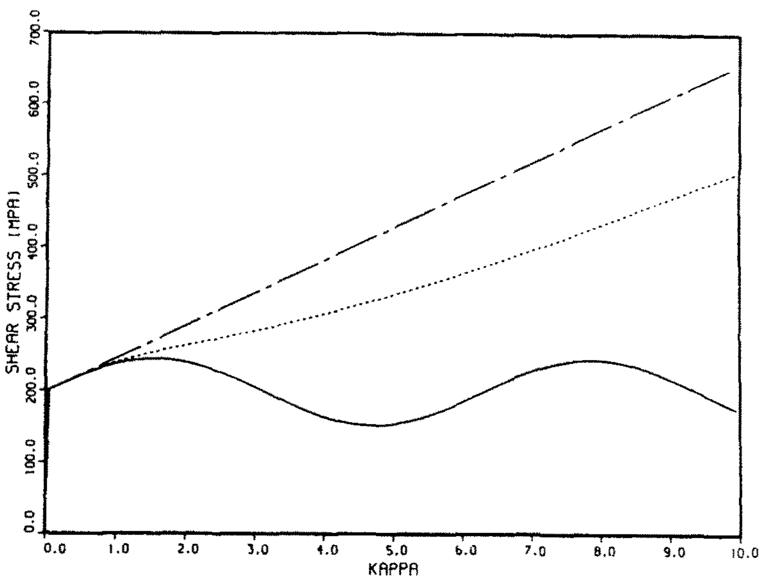

Fig. 1. Shear stress vs shear strain for the solution to the simple shear problem. The top curve is the solution obtained using an isotropic hardening model. The dotted line is the solution using a kinematic hardening model employing the Green–Naghdi stress rate and the solid line is the solution using a kinematic hardening model employing the Jaumann stress rate.

hardening using the Green–Naghdi rate lies between these two and has a monotonically increasing shear stress.

On the basis of this simple example alone, we cannot make sweeping statements regarding the general utility of one stress rate versus another. It does point out, though, some interesting features which bear examination. First, it indicates that the use of the Jaumann rate in the simple constant coefficient case can lead to results which are clearly not physically realistic (the period of the sinusoidal response is independent of material constants). Thus, an identification of the parameters of the constitutive equation based on a uniaxial tension test alone may not given meaningful results under general loading. The use of the Jaumann rate is not necessarily ruled out, but if the problem with the simple shear response is to be corrected, it will be by introducing additional complexity into the form of the constitutive relation, (perhaps by letting $q(\mathbf{D})$ in eqn (54) be a function of $\mathbf{T}$ as in the general hypoelastic constitutive relation, eqn (13)). The problem then becomes one of identifying the additional coefficients associated with such a generalization.

In discussing the Green–Naghdi rate, we note that the shear stress is not a linear function of the shear strain. It is, though, an increasing function somewhat below the curve for isotropic hardening. This response is in qualitative agreement with experimental observations that when uniaxial and torsion data are compared to one another through use of the Von Mises effective stress, the torsion data is below the uniaxial data and the difference increases with increasing deformation. A recent discussion of this may be found in [24].

In addition to this qualitative observation, there are several factors which may make the use of this rate a desirable choice. As already mentioned, it is obtained from the rate of change of a quantity which is both a measure of true stress and is invariant. Its derivation makes use of the intermediate configuration $\bar{\mathcal{B}}$ which allows the description of a constitutive equation which is independent of the local rotation $\mathbf{R}$. That is, use of this rate is associated with the idea that the relation between stress and deformation should depend only upon the motion causing the stretching of material elements and not on the local rotation of such elements. Also, since $\boldsymbol{\sigma}$ is invariant under superposed rigid body motions, there is no restriction of isotropy as when the Cauchy stress $\mathbf{T}$ is used.

In summary, we have shown that the oscillatory response observed in [1] for the simple shear problem with kinematic hardening is caused by the use of the Jaumann rate of stress in a constitutive model which predicts linear hardening in uniaxial tension. We have also shown that an alternative theory which makes use of the polar decomposition[14] leads to a monotonically increasing shear stress with increasing shear strain. In addition, the fact that this theory employs quantities which are invariant measures of the current configuration may make it more useful in modelling the plastic response of actual materials.

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